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VIBRATION ANALYSIS OF ANISOTROPIC OPEN CYLINDRICAL SHELLS SUBJECTED TO A FLOWING FLUID

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A theory is presented for the determination of the effects of a flowing fluid on the vibration characteristics of an open, anisotropic cylindrical shell submerged and subjected simultaneously to an internal and external flow. The case of an open shell partially or completely filled with liquid is also investigated. The structure may be uniform or nonuniform in the circumferential direction. The formulation used is a combination of finite element method and classical shell theory. The displacement functions are derived from exact solutions of Sanders' shell equations. The velocity potential and Bernoulli's equation for a liquid finite element yield an expression for fluid pressure as a function of the nodal displacements of the element and three forces (inertial, centrifugal and Coriolis) of the moving fluid. An analytical integration of the fluid pressure over the liquid element leads to three components: mass, stiffness and damping matrices. Calculations are given to illustrate the dynamic behaviour of open and closed cylindrical shells subjected to a flowing fluid, as well as shells partially or completely filled with liquid. Reasonable agreement is found with other theories and experiments.

1. INTRODUCTION

KNOWLEDGE OF THE VIBRATION CHARACTERISTICS of fluid-filled cylindrical shells and panels is of considerable practical interest, since cylindrical shells and panels are commonly used to contain or convey fluids. There are many ways in which the presence of the fluid may influence the dynamics of the structure. If the structure contains a stationary gas at low pressure, then the vibration of the shell differs only slightly from that of the same shell *in vacuo*. If the fluid is compressible, the compressibility of the fluid alters the effective stiffness of the system. Also, if the density of the fluid is relatively high, as in the case of a liquid, then the fluid exerts considerable inertial loading on the shell, and this results in a significant lowering of the resonant frequencies. Other effects of coupled fluid–shell motions occur when the fluid is flowing. Depending upon the boundary conditions, if the flow velocities are high, buckling or oscillatory flexural instabilities are possible.

The dynamics of coupled fluid–shells were reviewed extensively by Brown (1982), Au-Yang (1986), Paidoussis & Li (1993) and others (Mistry & Menenzes 1995; Harari *et al.* 1994; Cheng 1994; Han & Liu 1994; Terhune & Karim-Panahi 1993; Brenneman & Au-Yang 1992; Endo & Tosaka 1989 and Goncalves & Batista 1987). There have been few analyses of closed cylindrical shells having axially varying thickness. Similarly, while there is extensive literature relevant to the vibration of empty open cylindrical shells (cylindrical panels), no analysis has been found of open cylindrical shells,

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circumferentially nonuniform, totally submerged and subjected simultaneously to an internal and external flow.

The purpose of this study is to present a method for the dynamic and static analysis of open, thin, anisotropic cylindrical shells containing flowing fluid. The structure may be uniform or nonuniform in the circumferential direction and we consider the problem of open cylindrical shells which are freely simply supported ($V = W = 0$) along their curved edges and have arbitrary straight-edge boundary conditions.

The method is a hybrid of the finite element method, and classical shell and fluid-dynamics theories. The structure is subdivided into cylindrical panel-segment finite elements. The displacement functions are derived from Sanders' (1959) equation of thin cylindrical shells. In this approach, it is possible to determine the mass and stiffness matrices of the individual finite elements by exact analytical integration. Accordingly, this method is more accurate than the more usual finite element methods based on polynomial displacement functions.

To account for the fluid effect on the structure, a panel finite-fluid element bounded by two nodal lines is considered. By solving the equations of motion for the fluid element, an expression for fluid pressure as a function of the displacements of the element is obtained. Analytical integration for the pressure distribution along the element yields three components: the mass, stiffness and damping matrices for a fluid element. Global matrices are, then, obtained by superimposing the individual matrices. The eigenvalue and eigenvector problem is solved by means of the equation reduction technique.

The hybrid approach (finite element—shell theory—fluid theory) has been applied with satisfactory results to the dynamic linear and nonlinear analysis of cylindrical (Lakis & Païdoussis 1971, 1972d, 1973; Lakis 1976a, b; Lakis, Sami & Rousselet 1978; Lakis & Laveau 1991 and Lakis & Sinno 1992), conical (Lakis, Van Dyke & Ouriche 1992), spherical (Lakis, Tuy & Selmane 1989), isotropic and anisotropic, uniform and axially nonuniform shells both empty and liquid filled. This method has been applied also to the dynamic analysis of circular and annular plates (Lakis & Selmane 1990a, b) and to an open anisotropic and circumferentially nonuniform cylindrical shell (Selmane & Lakis 1995). This study is an attempt to determine the vibration of a circumferentially nonuniform open cylindrical shell, subjected to a flowing fluid. The case of an open cylindrical shell partially or completely filled with liquid is also studied.

2. DETERMINATION OF THE DISPLACEMENT FUNCTIONS

Sanders' (1959) equations for thin, cylindrical shells, in terms of axial, tangential and radial displacements (U, V, W) of the mean surface of the shell (Figure 1) and in terms of element P_{ij} of the anisotropic matrix of elasticity $[P]$ are

$$L_1(U, V, W, P_{ij}) = 0, \qquad L_2(U, V, W, P_{ij}) = 0, \qquad L_3(U, V, W, P_{ij}) = 0, \tag{1}$$

where L_k ($k = 1, 2, 3$) are three linear differential operators, the form of which is given fully in Appendix A.

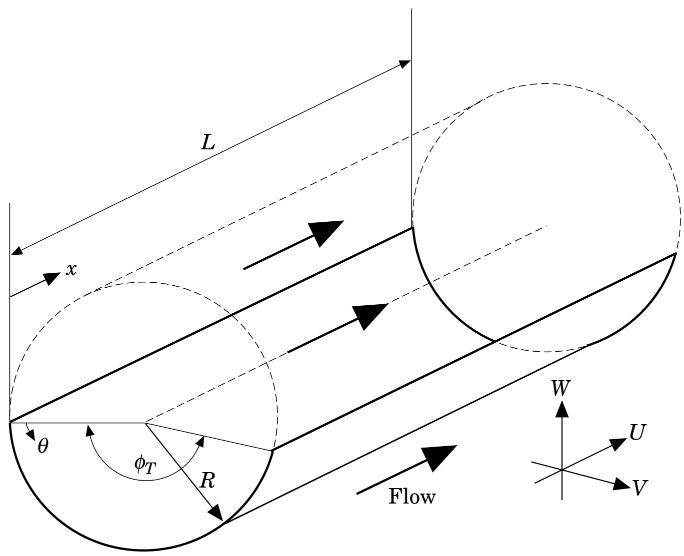


Figure 1. Open cylindrical shell geometry.

The strain-displacement relation is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ 2\bar{\varepsilon}_{x\theta} \\ \kappa_x \\ \kappa_\theta \\ 2\bar{\kappa}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\dfrac{\partial V}{\partial \theta} + \dfrac{W}{R} \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{R^2}\dfrac{\partial^2 W}{\partial \theta^2} + \dfrac{1}{R^2}\dfrac{\partial V}{\partial \theta} \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x\,\partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial \theta} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{Bmatrix}. \tag{2}$$

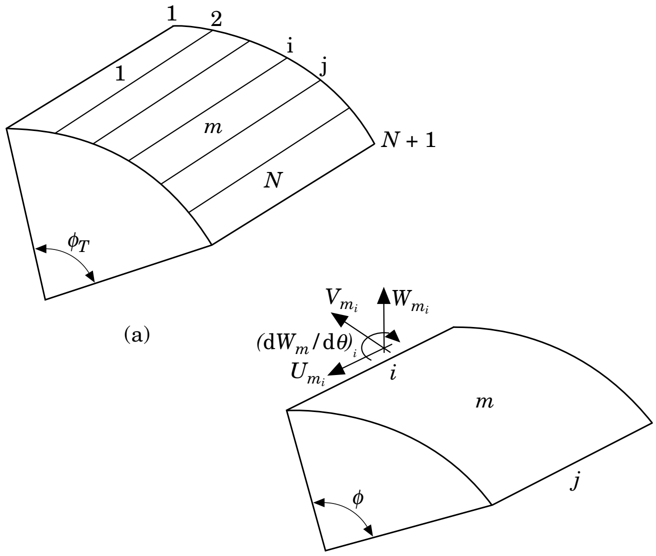
The finite element used is shown in Figure 2. It is a cylindrical panel segment defined by two line-nodes, *i* and *j*. Each node has four degrees of freedom: three displacements (axial, circumferential and radial) and one rotation. The panels are assumed to be freely simply supported ($V = W = 0$) along their curved edges and to have arbitrary straight edge boundary conditions.

For motions associated with the mth axial wave number, we may write

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = \begin{bmatrix} \cos m\pi x/L & 0 & 0 \\ 0 & \sin m\pi x/L & 0 \\ 0 & 0 & \sin m\pi x/L \end{bmatrix} \begin{Bmatrix} U_m(\theta) \\ W_m(\theta) \\ V_m(\theta) \end{Bmatrix} = [T_m] \begin{Bmatrix} U_m(\theta) \\ W_m(\theta) \\ V_m(\theta) \end{Bmatrix}. \tag{3}$$

By substituting equation (3) into equation (1) and letting

$$U_m(\theta) = A\mathrm{e}^{\eta\theta}, \qquad V_m(\theta) = B\mathrm{e}^{\eta\theta}, \qquad W_m(\theta) = C\mathrm{e}^{\eta\theta}, \tag{4}$$



(b)

Figure 2. (a) Finite element idealization; (b) nodal displacements at node i. N is the number of finite elements.

we obtain

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m][R]\{C\}, \tag{5}$$

where $[R]$ is a (3×8) matrix given by

$$R(1,j) = \alpha_j \mathrm{e}^{\eta_j \theta}, \qquad R(2,j) = \mathrm{e}^{\eta_j \theta}, \qquad R(3,j) = \beta_j \mathrm{e}^{\eta_j \theta}; \qquad j = 1, \ldots, 8; \tag{6}$$

η_j $(j = 1, \ldots, 8)$ are the roots of the characteristic equation of the empty panel. As A, B and C are not independent, we may write $A = \alpha C$ and $B = \beta C$, which determine α_j and β_j. $\{C\}$ is a vector of eight constants which are linear combinations of the C_j. The eight C_j are the only free constants, which must be determined from eight boundary conditions, four at each straight edge of the finite element.

We now express the nodal displacement vectors as follows:

$$\{\delta_i\} = \left\{ U_{mi},\, W_{mi},\, \left(\frac{\mathrm{d}W_m}{\mathrm{d}\theta}\right)_i,\, V_{mi} \right\}^{\mathrm{T}}. \tag{7}$$

Each $\{\delta_i\}$ may be determined from equation (5), where θ in $[R]$ now has a definite value, $\theta = 0$ or $\theta = \phi$, as the case may be; hence we obtain

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [A]\{C\}, \tag{8}$$

where the elements of matrix $[A]$ are determined from those of matrix $[R]$ and given by:

$$\begin{aligned} &A(1,j) = \alpha_j, && A(2,j) = 1, && A(3,j) = \eta_j, \\ &A(4,j) = \beta_j, && A(5,j) = \alpha_j \mathrm{e}^{\eta_j \phi}, && A(6,j) = \mathrm{e}^{\eta_j \phi}, \\ &A(7,j) = \eta_j \mathrm{e}^{\eta_j \phi}, && A(8,j) = \beta_j \mathrm{e}^{\eta_j \phi}; && j = 1, \ldots, 8 \end{aligned} \tag{9}$$

Finally, combining equation (5) and (8), we obtain

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m][R][A^{-1}]\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N]\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}, \tag{10}$$

which defines the displacement functions.

3. MASS AND STIFFNESS MATRICES FOR EMPTY FINITE ELEMENTS

The strains are related to the displacements through equations (2); accordingly, we may now express $\{\varepsilon\}$ in terms of δ_i and δ_j, and after lengthy manipulations we obtain

$$\{\varepsilon\} = \begin{bmatrix} [T_m] & 0 \\ 0 & [T_m] \end{bmatrix}[Q][A^{-1}]\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B]\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}, \tag{11}$$

where $[Q]$ is a (6×8) matrix given in Appendix B.

The corresponding stresses may be related to the strains by the elasticity matrix $[P]$:

$$\{\sigma\} = [P]\{\varepsilon\} = [P][B]\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}. \tag{12}$$

The matrix P of an anisotropic shell is given by

$$[P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix}. \tag{13}$$

The elements p_{ij} of $[P]$ characterize the shell anisotropy, which depends on the mechanical properties of the material of the structure.

The mass and stiffness matrices, $[m_s]$ and $[k_s]$ respectively, for one finite element may be written as follows:

$$[m_s] = \rho_s t \int_0^L \int_0^\phi [N]^T[N]\,\mathrm{d}A \quad \text{and} \quad [k_s] = \int_0^L \int_0^\phi [B]^T[P][B]\,\mathrm{d}A, \tag{14}$$

where ρ_s is the density of the shell, t its thickness, $\mathrm{d}A$ a surface element, $[P]$ the elasticity matrix and the matrices $[N]$ and $[B]$ are obtained from equations (10) and (11), respectively.

The matrices $[m_s]$ and $[k_s]$ were obtained analytically by carrying out the necessary matrix operations and integration over x and θ in equation (14). The global matrices $[M_s]$ and $[K_s]$ may be obtained, respectively, by superimposing the mass $[m_s]$ and stiffness $[k_s]$ matrices for each individual panel finite element. See (Selmane & Lakis 1995) for more details.

4. BEHAVIOUR OF THE FLUID-SHELL INTERACTION

4.1. Equations of Motion

The dynamic behaviour of an open shell subjected to a pressure field can be represented by the following system:

$$[[M_s]-[M_f]]\{\ddot{\delta}\} - [C_f]\{\dot{\delta}\} + [[K_s]-[K_f]]\{\delta\} = \{F\}, \tag{15}$$

where $\{\delta\}$ is the displacement vector, $[M_s]$ and $[K_s]$ are, respectively, the mass and stiffness matrices of the system *in vacuo*; $[M_f]$ and $[C_f]$ and $[K_f]$ represent the inertial, Coriolis and centrifugal forces of the liquid flow, and $\{F\}$ represents the external forces.

4.2. Assumptions

We assume here that the structure is subjected only to potential flow which induces inertial, Coriolis and centrifugal forces to participate in the vibration pattern. These forces are coupled with the elastic deformation of the shell.

The mathematical model which is developed is based on the following hypotheses: (i) the fluid flow is potential; (ii) vibration is linear (small deformations); (iii) since the flow is inviscid, there is no shear and the fluid pressure is purely normal to the shell wall; (iv) the fluid mean velocity distribution is assumed to be constant across a shell section; and (v) the fluid is incompressible.

4.3. Mass, Stiffness and Damping Matrices of the Moving Fluid

With the assumptions of Section 4.2, the velocity potential must satisfy the Laplace equation. This relation is expressed in the cylindrical coordinate system by

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial x^2} = 0; \tag{16}$$

Φ is the potential function that represents the velocity potential. Therefore, we have

$$V_x = U_x + \frac{\partial\Phi}{\partial x}, \qquad V_\theta = \frac{1}{R}\frac{\partial\Phi}{\partial\theta}, \qquad V_r = \frac{\partial\Phi}{\partial r}, \tag{17}$$

where U_x is the mean velocity along the shell in the x-direction. The remaining components of velocity are disturbance or perturbation velocity components; V_x, V_θ and V_r are respectively the axial, tangential and radial components of the fluid velocity.

The Bernouilli equation is given by:

$$\left.\frac{\partial\Phi}{\partial t} + \frac{1}{2}V^2 + \frac{P}{\rho_f}\right|_{r=\xi} = 0. \tag{18}$$

Introducing equation (17) into equation (18) and taking into account only the linear terms, we find the dynamic pressure, P:

$$P_u = \rho_{fu}\left.\left\{\frac{\partial\Phi_u}{\partial t} + U_{xu}\frac{\partial\Phi_u}{\partial x}\right\}\right|_{r=\xi}, \tag{19}$$

in which the subscript u represents "internal" or "external" fluid as the case may be:

$$\text{if } u = i \quad \text{then } \xi = R_i = R - \tfrac{1}{2}t; \tag{20}$$

$$\text{if } u = e \quad \text{then } \xi = R_e = R + \tfrac{1}{2}t. \tag{21}$$

A full definition of the flow requires that a condition be applied to the structure–fluid interface. The impermeability condition ensures contact between the shell and the fluid. This should be

$$V_r\big|_{r=R} = \left.\frac{\partial\Phi}{\partial r}\right|_{r=R} = \left.\left(\frac{\partial W}{\partial t} + U_x\frac{\partial W}{\partial x} + \tfrac{1}{2}U_x^2\frac{\partial^2 W}{\partial x^2}\right)\right|_{r=R}. \tag{22}$$

From the theory of shells [see equation (5)], we have

$$W(x,\theta,t) = \sum_{j=1}^{8} C_j e^{\eta_j\theta}\sin\frac{m\pi x}{L}e^{i\omega t}. \tag{23}$$

Assuming then,

$$\Phi(x, \theta, r, t) = \sum_{j=1}^{8} R_j(r) S_j(x, \theta, t) \tag{24}$$

and applying the impermeability condition, equation (22), with the radial displacement given by relation (23), we determine the function $S_j(x, \theta, t)$. Introducing this explicit term $S_j(x, \theta, t)$ into equation (24) and then into equation (19), we find a relation for the dynamic pressure as a function of the displacement W_j and the function $R_j(r)$:

$$P_u = -\rho_f \sum_{j=1}^{8} \frac{R_j(r)}{R_j^{\cdot}(R)} \left[\ddot{W}_j + 2U_{xu}\dot{W}_j' + \frac{U_{xu}^2}{2}\dot{W}_j'' + U_{xu}^2 W_j'' + \frac{U_{xu}^3}{2} W_j''' \right] \tag{25}$$

where ( )', ( )$^{\cdot}$ and ( )$'$ represent $\partial(\ )/\partial r$, $\partial(\ )/\partial t$ and $\partial(\ )/\partial x$, respectively.

By using relation (16), we obtain the following Bessel equation:

$$r^2 \frac{\mathrm{d}^2 R_j(r)}{\mathrm{d}r^2} + r\frac{\mathrm{d}R_j(r)}{\mathrm{d}r} + R_j(r)\left[\left(\frac{im\pi}{L}\right)^2 r^2 - (\mathrm{i}\eta_j)^2\right] = 0, \tag{26}$$

where i is the complex number, $\mathrm{i}^2 = -1$, and η_j is the complex solution of the characteristic equation.

The general solution of equation (26) is given by:

$$R_j(r) = A\mathrm{J}_{i\eta_j}\left(\frac{im\pi}{L} r\right) + B\mathrm{Y}_{i\eta_j}\left(\frac{im\pi}{L} r\right) \tag{27}$$

where $J_{i\eta j}$ and $Y_{i\eta j}$ are, respectively, the Bessel functions of the first and second kind of order i_{η_j}.

For internal flow, the solution (27) must be finite on the axis of the shell ($r = 0$); this means we have to set the constant B equal to zero. For external flow ($r \to \infty$); this means that the constant A is equal to zero. When the shell is simultaneously subjected to internal and external flow, we have to take the complete solution (27).

Finally, we obtain the equation for the pressure on the wall as follows:

$$P_u = -\rho_u \sum_{j=1}^{8} Z_{uj}\left(\frac{im\pi R_u}{L}\right)\left[\ddot{W}_j + 2U_{xu}\dot{W}_j' + \frac{U_{xu}^2}{2}\dot{W}_j'' + U_{xu}^2 W_j'' + \frac{U_{xu}^3}{2} W_j''' \right] \tag{28}$$

where ( )$^{\cdot}$ and ( )$'$ represent $\partial(\ )/\partial t$ and $\partial(\ )/\partial x$, respectively, and

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\dfrac{\mathrm{J}_{i\eta_j+1}(im\pi R_u/L)}{\mathrm{J}_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = i, \tag{29}$$

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\dfrac{\mathrm{Y}_{i\eta_j+1}(im\pi R_u/L)}{\mathrm{Y}_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = e, \tag{30}$$

where η_j ($j = 1, \ldots, 8$) are the roots of the characteristic equation of the empty shell; $\mathrm{J}_{i\eta j}$ and $\mathrm{Y}_{i\eta j}$ are, respectively, the Bessel functions of the first and second kind of order i_{η_j}; m is the axial mode number, R the mean radius of the shell, and L its length; the subscript u is equal to i for internal flow and is equal to e for external flow.

By introducing the displacement function (10), into the dynamic pressure expression (28) and performing the matrix operation required by the finite element method, the mass, damping and stiffness matrices for fluid are obtained by evaluating the following integral:

$$\int_A [N]^T \{P_u\}\,\mathrm{d}A, \tag{31}$$

leading to

$$[m_f] = [A^{-1}]^T[S_f][A^{-1}], \qquad [c_f] = [A^{-1}]^T[D_f][A^{-1}], \tag{32, 33}$$

$$[k_f] = [A^{-1}]^T[G_f][A^{-1}]. \tag{34}$$

The matrix $[A]$ is given by equation (9) and the elements of $[S_f]$, $[D_f]$ and $[G_f]$ are given by

$$S_f(r,s) = -\frac{RL}{2} I_{rs}(\rho_i Z_{is} - \rho_e Z_{es}), \tag{35}$$

$$D_f(r,s) = \frac{Rm^2\pi^2}{4L} I_{rs}(\rho_i U_{xi}^2 Z_{is} - \rho_e U_{xe}^2 Z_{es}), \tag{36}$$

$$G_f(r,s) = \frac{Rm^2\pi^2}{2L} I_{rs}(\rho_i U_{xi}^2 Z_{is} - \rho_e U_{xe}^2 Z_{es}), \tag{37}$$

where r, $s = 1, \ldots, 8$; ρ is the density of the fluid, and U_x its velocity; Z is defined by relations (29) and (30); the subscript i means internal flow and e means external flow; and I_{rs} is defined by

$$\begin{aligned} I_{rs} &= \frac{1}{(\eta_r + \eta_s)}[\mathrm{e}^{(\eta_r+\eta_s)\phi} - 1) \qquad &&\text{for } \eta_r + \eta_s \neq 0, \\ I_{rs} &= \phi &&\text{for } \eta_r + \eta_s = 0, \end{aligned} \tag{38}$$

in which $r, s = 1, \ldots 8$; η is the root of the characteristic equation of the empty shell and ϕ is the angle for one finite element.

Finally, the global matrices $[M_f]$, $[C_f]$ and $[K_f]$ may be obtained, respectively, by superimposing the mass $[m_f]$, damping $[c_f]$ and stiffness $[k_f]$ matrices for each individual fluid finite element.

5. EIGENVALUE AND EIGENVECTOR PROBLEM

The eigenvalue and eigenvector problem is solved by means of the equation reduction technique. Equation (15) may be rewritten as follows:

$$\begin{bmatrix} [0] & \dfrac{1}{\omega_0}[M] \\ \dfrac{1}{\omega_0^2}[M] & \dfrac{1}{\omega_0}[C] \end{bmatrix} \begin{Bmatrix} \ddot{\delta} \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -\dfrac{1}{\omega_0}[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \dot{\delta} \\ \delta \end{Bmatrix} = \{0\}, \tag{39}$$

where

$$[M] = [M_s] - [M_f], \qquad [K] = [K_s] - [K_f], \qquad [C] = [C_f]; \tag{40}$$

$[M_s]$ and $[K_s]$ are the global mass and stiffness matrices for the empty shell, $[M_f]$, $[C_f]$ and $[K_f]$ are the global mass, damping and stiffness matrices for the fluid; $\omega_0 = p_{11}$ is the first element of the elasticity matrix.

The eigenvalue problem is given by

$$|[DD] - \Lambda[I]| = 0, \tag{41}$$

where

$$[DD] = \begin{bmatrix} [0] & [I] \\ -\dfrac{1}{\omega_0^2}[K]^{-1}[M] & -\dfrac{1}{\omega_0}[K]^{-1}[C] \end{bmatrix} \tag{42}$$

and $\Lambda = 1/(\omega_0^2\omega^2)$; ω is the radian natural frequency of the system.

If the velocity of the fluid ($U_x = 0$), the eigenvalue problem in this particular case may be reduced to

$$\left| \frac{1}{\omega_0^2} [K]^{-1}[M] - \Lambda[I] \right| = 0, \qquad (43)$$

and $\omega = 1/(\omega_0 \Lambda)$.

Matrices $[K]$, $[M]$ and $[C]$ are square matrices of order $\mathrm{NDF}(N+1) - J$, where NDF is the number of degrees of freedom at each node, N is the number of finite elements in the structure and J is the number of constraints applied.

6. CALCULATIONS AND DISCUSSION

Calculations have already been conducted to test the theory in the case of *empty* open and closed shells. The free vibrations of uniform and circumferentially nonuniform, isotropic and orthotropic open and closed shells were obtained for a variety of boundary conditions (Selmane & Lakis 1995). The computed natural frequencies were compared to those obtained by other theories and from experiments; the results were in agreement within a range of ±5%.

Here we present some calculations to test the theory in the case of *liquid-filled* open and closed cylindrical shells. In the case when the shell is subjected to flowing fluid, the dynamic stability of this type of problem is analysed.

6.1. Free Vibration of Closed and Open Cylindrical Shells Partially or Completely Filled with Liquid

6.1.1. *Shell completely filled with liquid*

(a) For the first set of calculations, we determine the frequency parameters (Ω) for different values of R/t and L/R for shells completely filled with liquid (internal).

The results obtained (with 10 elements) for $n = 1$ are given in Table 1 in the case of free simply supported shells. We conclude that, as a result of the lateral pressure exerted by the liquid on the structure, the frequency parameters (Ω) depend both on

Table 1

Vibration parameter (Ω) of cylindrical shells simply supported at both ends and filled with liquid; $n = 1$, $m = 1$, $\nu = 0{\cdot}3$, $\rho_i = 1000\,\mathrm{kg/m^3}$ and $\Omega = \omega R\sqrt{\rho(1-\nu^2)/E}$

L/r	R/t	20	50	100	200	Baron & Bleich (1954); all values of R/t
Empty	2·0	0·5775	0·5900	0·6067	0·5711	0·5728
Full	2·0	0·4196	0·3288	0·2629	0·1810	—
Empty	4·0	0·2572	0·2581	0·2594	0·2603	0·2569
Full	4·0	0·1809	0·1372	0·1065	0·07998	—
Empty	8·0	0·08744	0·08747	0·08752	0·08756	0·0874
Full	8·0	0·06020	0·04489	0·03424	0·02269	—
Empty	10·0	0·05911	0·05911	0·05913	0·05914	0·0592
Full	10·0	0·04044	0·03005	0·02283	0·01684	—

TABLE 2

Natural frequencies (Hz) of a simply supported closed cylindrical shell, both when empty and when completely filled with liquid

	Empty		Full (inside fluid)	
(m, n)	Present method	Experimental (Lindholm *et al.* 1962)	Present method	Experimental (Lindholm *et al.* 1962)
(1, 2)	1133	1150	376	375
(1, 3)	629	640	234	250
(1, 4)	655	688	270	300
(1, 5)	942	995	422	430
(1, 6)	1353	1430	651	680
(1, 7)	1853	1938	940	970
(2, 3)	2067	2070	784	813
(2, 4)	1368	1430	568	600
(2, 5)	1248	1313	561	625
(2, 6)	1489	1570	714	755
(2, 7)	1927	2050	978	1000

L/R and R/t, in contrast to the case of the empty shell, where the R/t ratio has only a slight effect upon the results.

(b) Next, calculations were made for a steel cylindrical shell, simply supported at both ends, empty or completely filled with liquid. The pertinent data are as follows:

$$R = 37{\cdot}7\ \text{mm}, \qquad t = 0{\cdot}229\ \text{mm}, \qquad L = 234\ \text{mm},\ \nu = 0{\cdot}29, \qquad \rho_i/\rho_s = 0{\cdot}128.$$

The effects of the inertial force were calculated by this theory assuming $U_x = 0$ in equations (35)–(37). Table 2 shows some frequencies computed by the present method and compared with experimental results (Lindholm, Kaña & Abramson 1962) in the case of a closed cylindrical shell, both empty and completely filled with liquid. The results obtained by the present method agree with experimental results to within 10%. The boundary conditions used by Lindholm *et al.* ($V = W = 0$) at both ends are similar to ours. The difference between our method and the experimental results of Lindholm *et al.* is between 0·5% and 10% for the first two axial modes.

6.1.2. *Shells partially filled with liquid*

Here, we consider the case of a shell partially filled with liquid without taking into account the effects of the free surface. No dynamic pressure is imposed on the liquid free surface and the superficial tension is also neglected. This may be justified as follows: the natural frequencies of the empty shells in the modes under consideration are likely to be high, whereas the natural frequencies of free surface phenomena are likely to be low, at least in the lowest modes; accordingly, coupling between the shell modes and the liquid free-surface modes may be expected to be weak. However, Lindholm *et al.* have found experimentally that there is a possibility of nonlinear coupling between the low frequency, free-surface modes and the shell modes, resulting in subharmonic excitation of the former while the shell itself is oscillating at high frequencies.

In the case of vertical cylindrical shells, partially filled with liquid, it has been determined, by comparing the results of Mistry & Menenzes (1995) with those based on the theory presented by Lakis & Païdoussis (1971), that these effects are negligible for low frequencies (less than 3%), but may go up to 30% for modes higher than seven ($m \geq 7$).

The principal cause of this phenomenon is attributed to the kinetic energy developed by the movement of the free surface; this effect diminishes the natural frequencies of the system as a function of the geometric and physical properties of the shell and of the axial and circumferential waves number. However, the potential energy due to the elevation of the wave has little effect on natural frequencies, because this energy is very small compared to the energy of deformation of the shell. On the other hand, the effect of free surface is negligible at higher values of circumferential mode number ($n \geq 7$). The difference between frequencies calculated by the two methods (with and without free-surface effects) is given by the difference of the kinetic energy developed by the free surface of the fluid versus the kinetic energy developed by lateral surface of the fluid.

In the case of horizontal shells partially filled with liquid, the results might be different due to a larger free surface. Nevertheless, our results presented here are an indication of the dynamic behaviour of such a system. A paper, under preparation, will consider in detail the effects of the free surface in the case of both horizontal and vertical shells partially filled with liquid.

In the case of a closed cylindrical shell, Figures 3 and 4 show some frequencies computed by the present method in which the liquid level was varied from zero to full, in a cylinder with a horizontal axis.

We see that, for some modes, the frequency decreases rapidly with increasing d_1/d in the range $0 < d_1/d < 1/4$ approximately, and then decreases only slightly for higher fractional fillings. For other modes, however, the frequencies decrease appreciably with increasing d_1/d over the whole range of d_1/d, as might be expected.

Next, we present some results for an open cylindrical shell partially or completely filled with liquid. The open cylindrical shell is constructed of steel, is filled with water, and is simply supported at its four edges. The pertinent data are as follows (see Figure 1):

$$\phi_T = 180°, \qquad R = 37{\cdot}7\text{ mm}, \qquad t = 0{\cdot}229\text{ mm},$$
$$L = 234\text{ mm}, \qquad \nu = 0{\cdot}29, \qquad \rho_i/\rho_s = 0{\cdot}128,$$

In Figure 5(a), we see the behaviour of an open cylindrical shell, empty or filled with liquid, as a function of the number of circumferential modes. For a given m, the frequencies decrease to a minimum before they increase as the number of circumferential waves 'n' is increased. This behaviour was first observed for a shell *in vacuo* by Arnold & Warburton (1953), who were able to explain it by a consideration of the strain energy associated with bending and stretching of the reference surface. It may be concluded from their work, that at low n the bending strain energy is low and the stretching strain energy is high; while at the higher n, the relative contributions from the two types of strain energy are reversed. The interchange in the relative contributions of the bending and stretching strain energy as the circumferential wave number n is increased explains the decrease and subsequent increase in the natural frequencies indicated in Figure 5(a). An open cylindrical shell partially or completely filled with liquid will behave in the same way.

Figure 5(b) presents the eigenvectors in the case of incomplete shells. A closed cylinder, or any other shell undergoing vibrations, may be deformed in a variety of ways, as shown in Figure 3 where several configurations are given. Viewed from one end, the vibration of the cylinder may consist of any number of waves distributed around the circumference. Denoting the number of these waves by n, we see, in Figure 3, cases of n equal to 2, 3 and 4 (empty or completely filled shell). When viewed from

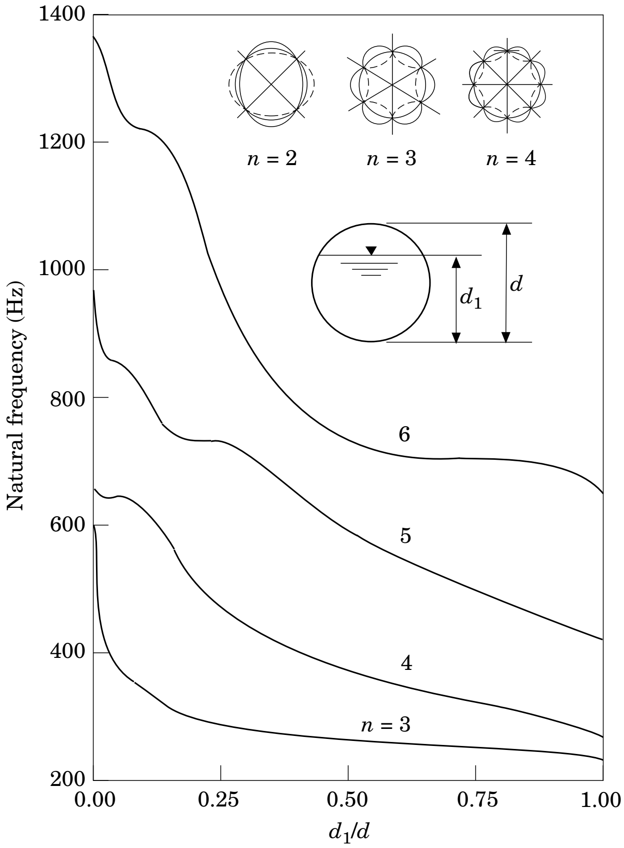


Figure 3. Natural frequencies of a partially filled closed cylindrical shell supported at both ends as a function of liquid level, $m = 1$; $R = 37{\cdot}7$ mm, $t = 0{\cdot}229$ mm, $L = 234$ mm, $\nu = 0{\cdot}29$, $\rho_l/\rho_s = 0{\cdot}128$.

its side, the deformation of the cylinder consists of a number of waves distributed along the length of a generator. We obtain for example, for $n = 4$, eight circumferential half-waves in the case of a complete shell (Figure 3) and four circumferential half-waves in the case of an incomplete shell [Figure 5(b)].

In the case of an open cylindrical shell partially filled with liquid, the curves of Figure 6 show a rapid decrease of the natural frequencies as a_1/a_2 increases from 0 to 3/4 approximately, and then decrease only slightly for higher fractional fillings.

To see the influence of the orientation of the shell, we present in Figure 7 the natural frequency as a function of the orientation of the shell and the free surface of the liquid (the liquid level: $a_1/a_2 = 0{\cdot}64$ at $\alpha = 0$, see Figure 6). We observe that the natural frequencies of the system decrease between the two extreme positions. The reduction is about 11% for the two modes ($m = 1$, $n = 2$) and ($m = 1$, $n = 7$).

6.2. Closed Orthotropic Cylindrical Shells Submerged in an Incompressible Fluid

In this calculation, we analyse the transverse vibration of isotropic and orthotropic cylindrical shells submerged in an incompressible fluid, simply supported at both ends. This case was analysed before by Ramachandran (1979) who used the Rayleigh-Ritz procedure. In Table 3, the values of the material properties used in the calculations are shown.

The natural frequencies of this shell-liquid system for $n = 4$ and 8, $m = 1$, $L/R = 2$

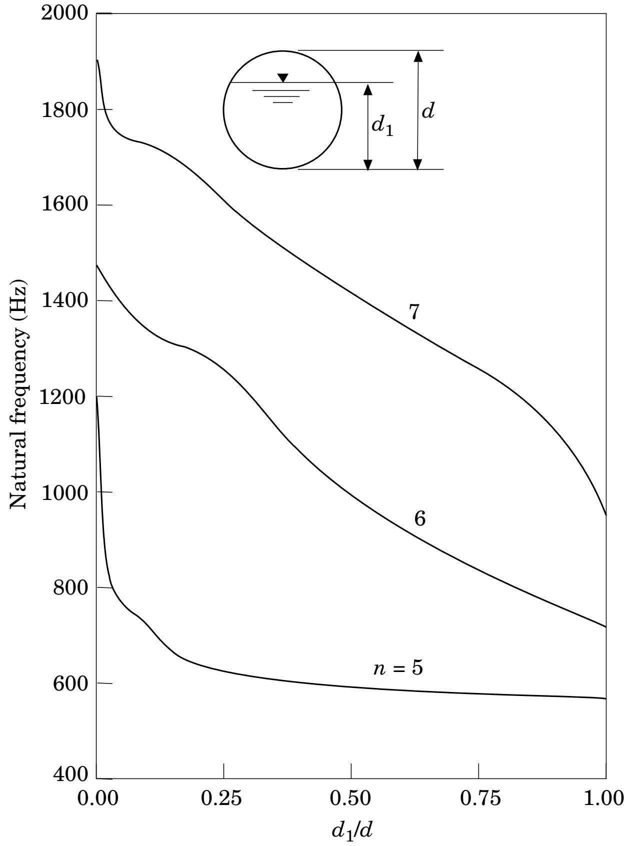


Figure 4. Natural frequencies of a partially filled closed cylindrical shell supported at both ends as a function of liquid level, $m = 2$; $R = 3{\cdot}7$ mm, $t = 0{\cdot}229$ mm, $L = 234$ mm, $\nu = 0{\cdot}29$, $\rho_i/\rho_s = 0{\cdot}128$.

and 4, and different material properties of the shell are given in Table 4. Four cases were studied, when the shell is empty; when the fluid is inside or outside the shell; and when the shell is submerged in a fluid. If we compare our results with those of Ramachandran (1979), we find that there is agreement within 6% in the case of the empty isotropic shell. In the case of the submerged isotropic shell (internal and external fluid), the agreement varies from 11 to 15%. However, when the material of the shell is orthotropic, we find big differences between the two models (in the order of 98%). On the other hand, in the case of the empty orthotropic cylindrical shell, our model has been tested (Selmane & Lakis 1995) and the results have been found to be within 5% of those of Leissa (1973).

Our model combines the advantages of finite element method which can deal with more complex shells (variable thickness, nonuniform materials, various boundary conditions, etc.), and the precision of formulation which the use of displacement functions derived from shell theory contributes (Lakis *et al.* 1992).

6.3. Dynamic Stability of Closed and Open Cylindrical Shells Subjected to a Flowing Fluid

6.3.1. *Closed cylindrical shell containing flowing fluid*

When the fluid is flowing, the shell is subjected to centrifugal, Coriolis and inertia forces. A simply supported shell with the following characteristics

$$L/R = 2, \qquad t/R = 0{\cdot}01, \qquad \rho_i/\rho_s = 0{\cdot}128, \qquad n = 5$$

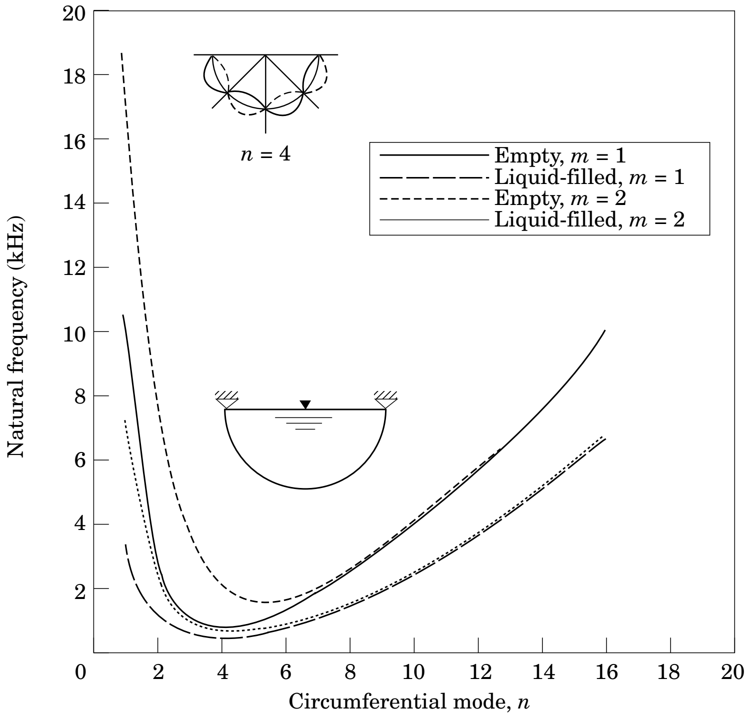


Figure 5(a). Natural frequencies of an empty and liquid-filled open cylindrical shell with $W = V = 0$ at the four edges as a function of circumferential mode number.

has been analysed, to see the influence of the speed of the flow U_{xi} on the frequencies (internal flow). The dimensionless parameters of frequency and velocity are $\bar{\omega} = \omega/\omega_0$ and $\bar{U} = U/U_0$, where

$$\omega_0 = \frac{\pi^2}{L^2}\left(\frac{K}{\rho_s t}\right)^{1/2}, \qquad K = \frac{Et^3}{12(1-\nu^2)}, \qquad U_0 = \frac{\pi^2}{L}\left(\frac{K}{\rho_s t}\right)^{1/2};$$

ω and U are respectively the natural frequency and the velocity of the flowing fluid.

The results are compared to a previous analysis by Weaver & Unny (1973) in Figure 8. We observe that the natural frequencies decrease with flow velocity. At zero flow

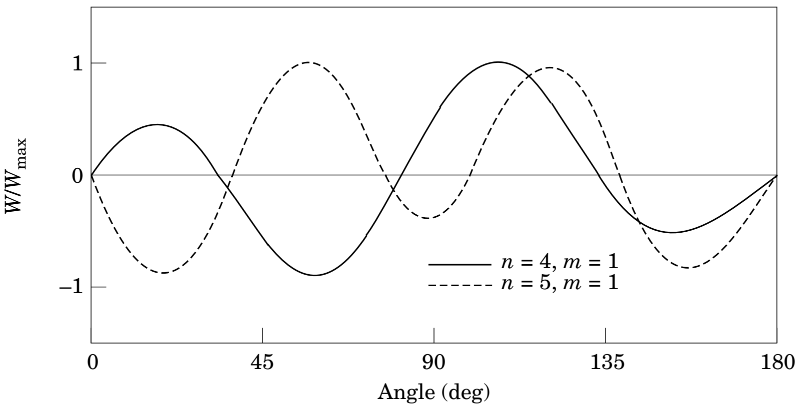


Figure 5(b). The circumferential shapes of a liquid-filled open cylindrical shell with $W = V = 0$ at the four edges for $n = 4, 5$ and $m = 1$.

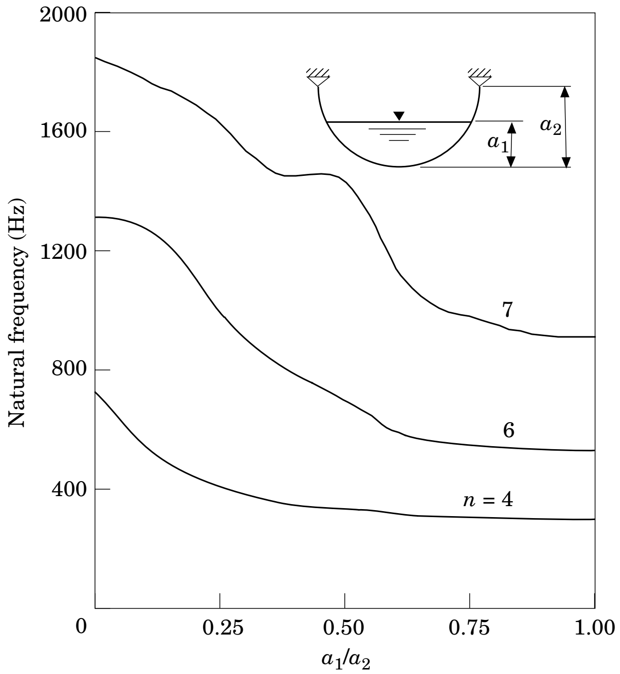


Figure 6. Natural frequencies of an empty and liquid-filled open cylindrical shell with $W = V = 0$ at the four edges as a function of liquid level; $m = 1$.

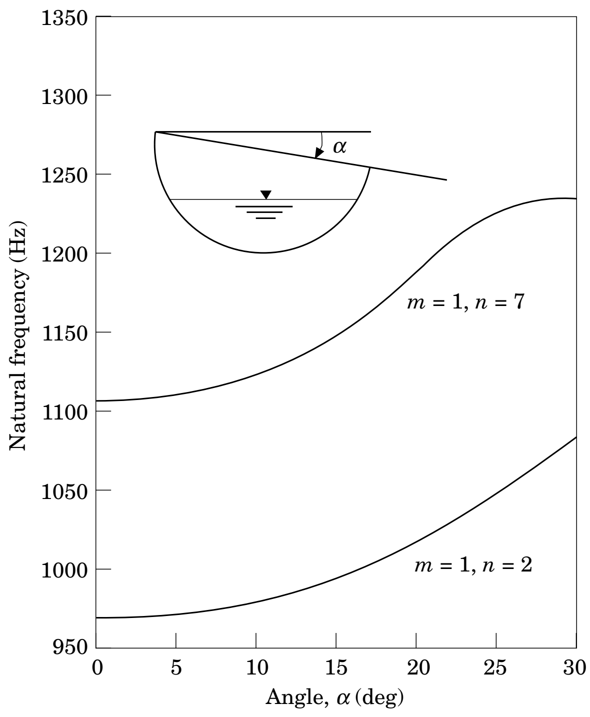


Figure 7. Natural frequencies of an empty and liquid-filled open cylindrical shell with $W = V = 0$ at the four edges as a function of the orientation of liquid level and the shell; $a_1/a_2 = 0{\cdot}64$ at $\alpha = 0$.

TABLE 3

Material and physical properties of the shell; $R = 0{\cdot}235$ m, $t = 0{\cdot}00235$ m, $\rho_s = 7850$ N/m^3, $\rho_f = 1000$ N/m^3

	E_x ($\times 10^{11}$ N/m^2)	E_θ ($\times 10^{11}$ N/m^2)	G ($\times 10^{11}$ N/m^2)	ν_x	ν_θ
Isotropy	21·981	21·981	0·8454	0·3	0·3
Orthotropy	1·0	0·5	0·1	0·05	0·025

TABLE 4

Frequency values (Hz) for simply supported cylindrical shells, empty and filled with liquid

Material	L/R	(n, m)	Theory	Empty	Inside and outside fluid (full)	Inside fluid	Outside fluid
Isotropic	4	(4, 1)	Present method	659	251·4	333·2	331·4
			Ramachandran (1979)	700	294·2	—	—
			Lakis (1976a)*	659	251·7	333·8	331·7
		(8, 1)	Present method	2187	1064	1361	1361
			Ramachandran (1979)	2200	944·1	—	—
			Lakis (1976a)*	2177	1073	1362	1360
Orthotropic	2	(4, 1)	Present method	240·1	92·2	121·9	121·6
			Ramachandran (1979)	—	183·1	—	—
			Lakis (1976a)*	238·8	92·4	121·7	121·9
		(8, 1)	Present method	327·3	158·5	203·3	200·2
			Ramachandran (1979)	—	248·5	—	—
			Lakis (1976a)*	324·1	160·7	203·2	203·9

* These results are computed from a computer program developed by Lakis and his co-workers and based on the theory presented in Lakis (1976a).

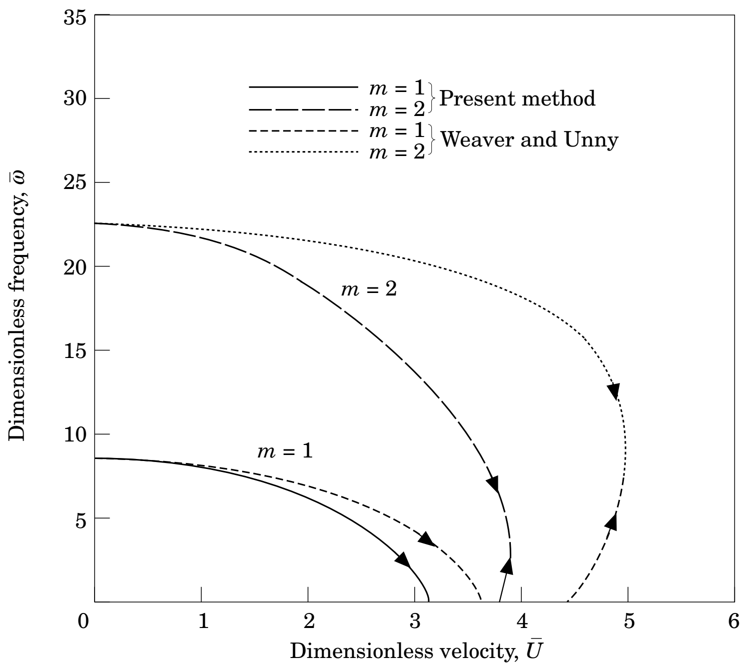


Figure 8. Stability of a simply supported closed cylindrical shell as a function of flow velocity, for internal flow; $L/R = 2$, $R/t = 100$, $\rho_i/\rho_s = 0{\cdot}128$, $n = 5$.

velocity, the two methods give the same results but, as the flow velocity increases the two term Galerkin method used by Weaver & Unny (1973) generates significantly different results from those of the present hybrid finite element method. This is due to the limitations associated with the use of too few terms in the application of Galerkin's method.

One of the most important criteria in determining the versatility of a method is the capacity to predict, with precision, both the high and low frequencies. Satisfaction of this criterion demands the use of a great many terms in Galerkin's method. The choice of the displacement functions which are derived from Sanders' (1959) classical shell theory enables our hybrid finite element model to give good values for the frequencies, low as well as high, with a small number of finite elements.

Our results predict that the first mode frequency becomes negative imaginary at $\bar{U} = 3{\cdot}1$, indicating static divergence instability in this mode. If the velocity is increased further, the first mode reappears and coalesces at $\bar{U} = 3{\cdot}95$ with that of the second mode to produce coupled mode flutter.

6.3.2. *Totally submerged open cylindrical shell subjected simultaneously to an internal and external flow*

An open cylindrical shell subjected simultaneously to an internal and external flow has been analysed. In this case there are no effects of the free surface because the shell is totally submerged in the flowing liquid. The data for the shell are as follows:

$$R/t = 165, \qquad L/R = 6{\cdot}2, \qquad \phi_T = 180° \qquad \rho_f/\rho_s = 0{\cdot}128, \qquad \nu = 0{\cdot}29.$$

We present here an examination of the natural frequencies of the system as functions of the flow velocity, and thereby a determination of the effect of flow on the dynamic behaviour of the system.

(a) *Simply supported—simply supported shell*

A simply supported open cylindrical shell containing flowing fluid (internal and external) has been analysed. Figure 9 shows the frequencies of the system as a function of the flow velocity. As the velocity increases from zero, the frequencies associated with all modes decrease; they remain real (the system being conservative) until, at sufficiently high velocities, they vanish, indicating the existence of a buckling-type (divergence) instability. At higher flow velocity the frequencies become purely imaginary. We predict the first loss of stability at a flow velocity equal to $\bar{U} = 7{\cdot}75$ for the mode ($m = 1, n = 4$).

(b) *Free–free shell*

The case of an open cylindrical shell having its straight edges free and the curved edges freely simply supported has been studied by means of the present theory. Figure 10 shows that natural frequencies associated with all modes decrease with increasing flow velocity until, at a value of $\bar{U} = 8{\cdot}5$, the system buckles in the ($m = 1$, $n = 6$) mode.

(c) *Clamped–clamped shell*

The calculations were performed for one open cylindrical shell having its straight edge clamped and the curved edges freely simply supported. Here, we study the influence of the flow velocity on the dynamic stability of the open shell containing internal and external flow. We observe in Figure 11 that the frequencies associated

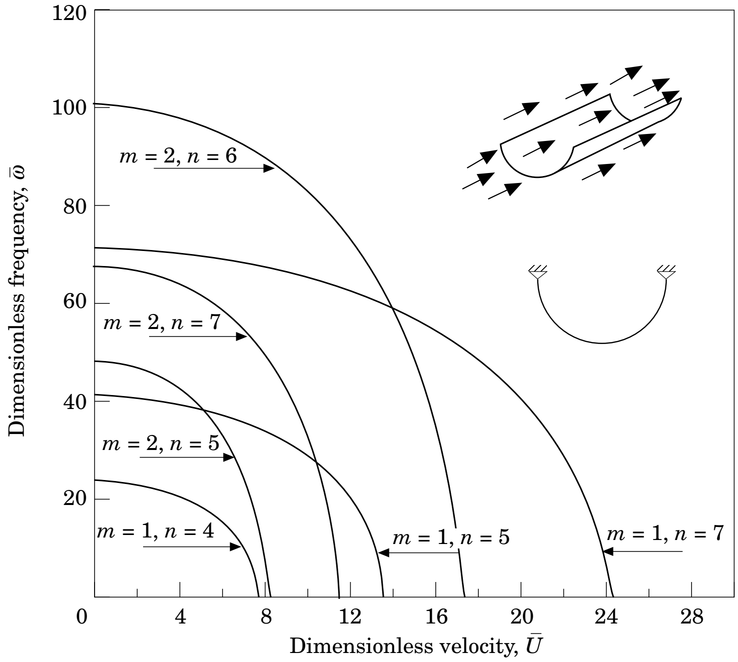


Figure 9. Stability of a simply supported submerged open cylindrical shell in a flowing fluid as a function of flow velocity.

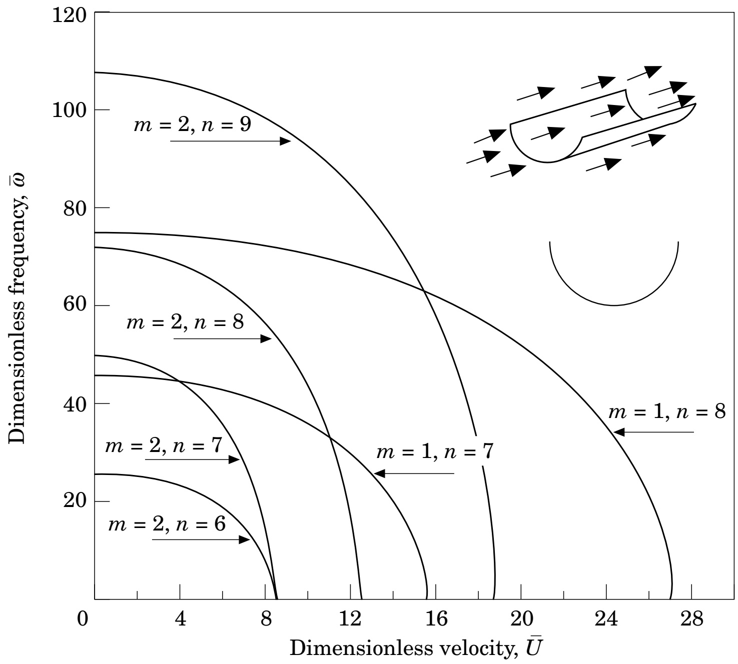


Figure 10. Stability of a free–free submerged open cylindrical shell in a flowing fluid as a function of flow velocity.

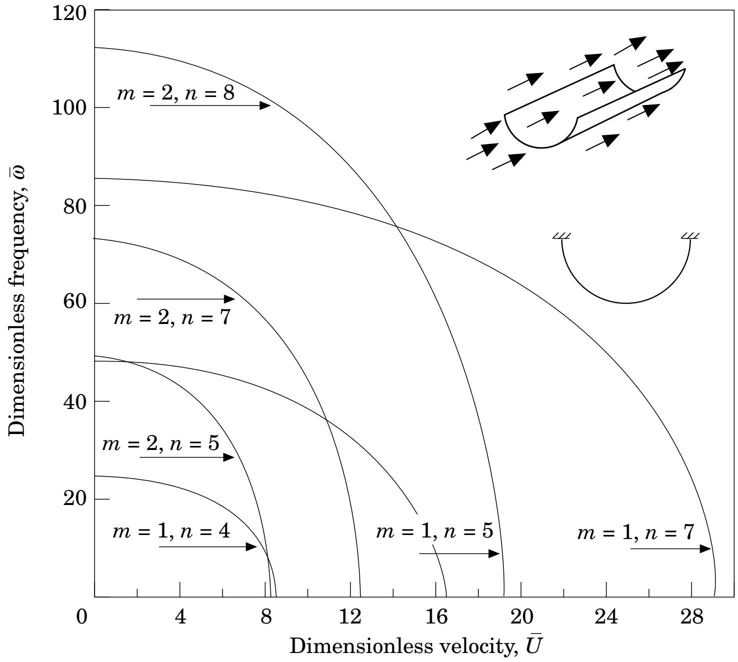


Figure 11. Stability of a clamped–clamped submerged open cylindrical shell in a flowing fluid as a function of flow velocity.

with all modes decrease with increasing flow velocity, and similarly to the case of simply supported–simply supported and free–free open shells, the frequencies remain real until at a sufficiently high velocity, they vanish, indicating the instability. For the stipulated boundary conditions, we predict the first loss of instability at $\bar{U} = 8{\cdot}25$ for the mode ($m = 1,\ n = 4$).

Finally, we observe in Figures 9, 10 and 11 that lower values of axial mode m given higher critical velocities.

(d) *Comparison between the boundary conditions*

In order to establish the effects of boundary conditions on the critical flow velocities which render the system dynamically unstable, we turn to Figure 12. We observe for the same mode and the same open shell with different boundary conditions, that the shell with free–free boundary conditions in its straight edges is the one which loses dynamic stability first.

For the mode ($m = 1,\ n = 7$) we have critical velocities as follows. For the free–free shell: $\bar{U} = 15{\cdot}5$; for the simply supported–simply supported shell: $\bar{U} = 24{\cdot}4$; and for the clamped–clamped shell: $\bar{U} = 29$. For the mode ($m = 2,\ n = 7$), we have respectively $\bar{U} = 8{\cdot}5$, 11·5, and 12.5.

7. CONCLUSIONS

The theory developed in this paper is used to predict the effects of inertia, Coriolis and centrifugal forces on the vibration characteristics of totally submerged anisotropic open and closed cylindrical shells, subjected simultaneously to an internal and external flow.

A cylindrical panel finite element was developed, making possible the derivation of

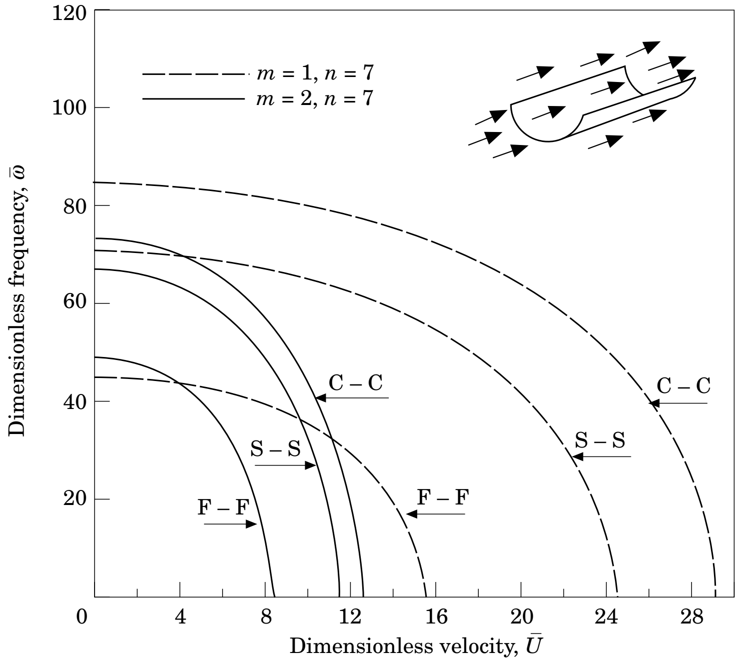


Figure 12. Effect of boundary conditions on the stability of an open cylindrical shell submerged in flowing fluid. F–F: Free–free; S–S Simply supported at the two ends; C–C: Clamped–clamped shell.

the displacement functions from the equations of motion of the shell. The mass and stiffness of each element were obtained by exact analytical integration.

The fluid pressure was derived from the velocity potential and from the linear impermeability and dynamic conditions applied to the shell–fluid interface. The finite element method was used to obtain the mass, stiffness and damping of fluid elements. The results obtained by this method were compared with other investigations, and satisfactory agreement was obtained. This method combines the advantages of finite element analysis which deals with complex shells, and the precision of formulation which the use of displacement functions derived from shell and fluid theories contributes.

This method enables us to predict the vibrational characteristics of circumferentially nonuniform open and closed cylindrical shells subjected to a flowing fluid. In addition, this theory may be applied to a curved plate subjected to a flowing fluid in the case of large values of the shell radius.

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APPENDIX A: EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin anisotropic cylindrical shell; they are the following [see equation (1)]:

$$L_1(U, V, W, p_{ij}) = p_{11}\frac{\partial^2 U}{\partial x^2} + \frac{p_{12}}{R}\left(\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{\partial W}{\partial x}\right) - p_{14}\frac{\partial^3 W}{\partial x^3} + \frac{p_{15}}{R^2}\left(\frac{\partial^3 W}{\partial x\,\partial\theta^2} + \frac{\partial^2 V}{\partial x\,\partial\theta}\right)$$
$$+\left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\left(\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2}\right) + \left(\frac{p_{36}}{R^2} - \frac{p_{66}}{2R^3}\right)\left(-\frac{2\partial^3 W}{\partial x\,\partial\theta^2} + \frac{3}{2}\frac{\partial^2 V}{\partial x\,\partial\theta} - \frac{1}{2}R\frac{\partial^2 U}{\partial\theta^2}\right);$$

$$L_2(U, V, W, p_{ij}) = \left(\frac{p_{21}}{R} + \frac{p_{51}}{R^2}\right)\left(\frac{\partial^2 U}{\partial x\,\partial\theta}\right) + \frac{1}{R}\left(\frac{p_{22}}{R} + \frac{p_{52}}{R^2}\right)\left(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\right)$$
$$-\left(\frac{p_{24}}{R} + \frac{p_{54}}{R^2}\right)\left(\frac{\partial^3 W}{\partial x^2\,\partial\theta}\right) + \frac{1}{R^2}\left(\frac{p_{25}}{R} + \frac{p_{55}}{R^2}\right)\left(-\frac{\partial^3 W}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}\right)$$
$$+\left(p_{33} + \frac{3p_{63}}{2R}\right)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 U}{R\,\partial x\,\partial\theta}\right) + \frac{1}{R}\left(p_{36} + \frac{3p_{66}}{2R}\right)\left(-2\frac{\partial^3 W}{\partial x^2\,\partial\theta} + \frac{3}{2}\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 U}{2R\,\partial x\,\partial\theta}\right);$$

$$L_3(U, V, W, p_{ij}) = p_{41}\frac{\partial^3 U}{\partial x^3} + \frac{p_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\,\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - p_{44}\frac{\partial^4 W}{\partial x^4} + \frac{p_{45}}{R^2}\left(-\frac{\partial^4 W}{\partial x^2\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right)$$
$$+\frac{2p_{63}}{R}\left(\frac{\partial^3 U}{R\,\partial x\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) + \left(\frac{2p_{66}}{R^2}\right)\left(-2\frac{\partial^4 W}{\partial x^2\,\partial\theta^2} + \frac{3}{2}\frac{\partial^3 V}{\partial x^2\,\partial\theta} - \frac{\partial^3 U}{2R\,\partial x\,\partial\theta^2}\right)$$
$$+\frac{p_{51}}{R^2}\frac{\partial^3 U}{\partial x\,\partial\theta^2} + \frac{p_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3} + \frac{\partial^2 W}{\partial\theta^2}\right) + \frac{p_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right) - \frac{p_{21}}{R}\frac{\partial U}{\partial x} - \frac{p_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\,\partial\theta^2}$$
$$-\frac{p_{22}}{R^2}\left(\frac{\partial V}{\partial\theta} + W\right) + \frac{p_{24}}{R}\frac{\partial^2 W}{\partial\theta^2} - \frac{p_{25}}{R^3}\left(-\frac{\partial^2 W}{\partial\theta^2} + \frac{\partial V}{\partial\theta}\right).$$

APPENDIX B: MATRIX $[Q]_{(6\times 8)}$

The elements of the matrix are given by

$$Q(1,j) = A_j e^{\eta_j\theta}, \qquad Q(4,j) = D_j e^{\eta_j\theta},$$
$$Q(2,j) = B_j e^{\eta_j\theta}, \qquad Q(5,j) = E_j e^{\eta_j\theta},$$
$$Q(3,j) = C_j e^{\eta_j\theta}, \qquad Q(6,j) = F_j e^{\eta_j\theta}.$$

The terms A_j, B_j, C_j, D_j, E_j and F_j ($j = 1, \ldots, 8$) may be expressed as follows:

$$A_j = -\frac{m\pi\alpha_j}{L}, \qquad B_j = -\frac{\eta_j\beta_j + 1}{R}, \qquad C_j = -\frac{m\pi\beta_j}{L} + \frac{\eta_j\alpha_j}{R},$$
$$D_j = -\frac{(m\pi)^2}{L^2}, \qquad E_j = -\frac{\eta_j^2 + \eta_j\beta_j}{R^2},$$

and

$$F_j = -\frac{2m\pi\eta_j}{RL} + \frac{3m\pi\beta_j}{2RL} - \frac{\eta_j\alpha_j}{2R^2}.$$

APPENDIX C: NOMENCLATURE

A, B, C	constants in equations defining U, V, W respectively
a_1/a_2	liquid-level ratio for an open cylindrical shell
c	velocity of sound in fluid
d_1/d	liquid level ratio for closed cylindrical shell
E	Young's modulus
e	exponential
i	$\mathrm{i}^2 = -1$
$\mathrm{J}_{i\eta j}$	Bessel function of the first kind and of order $\mathrm{i}\eta j$
K	bending stiffness, $Et^3/12(1-\nu^2)$
L	length of the shell
m	axial mode number
n	circumferential mode number
P_u	lateral pressure exerted on the shell, $u = i$ for internal pressure and $u = e$ for external pressure
p_{ij}	terms of elasticity matrix ($i = 1, \ldots, 6$; $j = 1, \ldots, 6$)
R	mean radius of the shell
R_j	solution of Bessel equation (27)
S_j	defined by equation (24)
t	thickness of the shell
U, V, W	axial, tangential and radial displacements
U_{xu}	velocity of the liquid
U_0	defined by $(\pi^2/L)(K/\rho_s t)^{1/2}$
$\bar{U}$	dimensionless velocity, U_{xu}/U_0
V_x, V_θ, V_r	axial, tangential and radial fluid velocity (17)
x	axial coordinate
$\mathrm{Y}_{i\eta j}$	Bessel function of the second kind and of order $i\eta j$
Z_{uj}	defined by equation (29) for $u = i$ and equation (30) for $u = e$
ε_x, ε_θ, $\bar{\varepsilon}_{x\theta}$	deformation of reference surface
η_i	complex roots of the characteristic equation
θ	circumferential coordinate
κ_x, κ_θ, $\bar{\kappa}_{x\theta}$	changes in curvature and torsion of reference surface
ν	Poisson's ratio
ρ_s	density of the shell material
ρ_f	density of fluid, $f = i$ for internal fluid and $f = e$ for external fluid
ϕ	angle for one finite element
ϕ_T	angle for the whole open shell
Φ	velocity potential
ω	natural frequency (rad/s)
ω_0	defined by $(\pi^2/L^2)(K/\rho_s t)^{1/2}$
$\bar{\omega}$	dimensionless frequency, ω/ω_0

List of matrices

$[A]$	defined by equation (9)
$[B]$	defined by equation (11)
$[c_f]$	damping matrix for a fluid finite element
$[C_f]$	damping matrix for the whole fluid
$\{C\}$	vector of arbitrary constants
$[D_f]$	defined by equation (36)
$[G_f]$	defined by equation (37)
$[k_f]$	stiffness matrix for a fluid finite element
$[k_s]$	stiffness matrix for a shell finite element
$[K_f]$	stiffness matrix for the whole fluid
$[K_s]$	stiffness matrix for the whole shell
$[m_f]$	mass matrix for a fluid finite element
$[m_s]$	mass matrix for a shell finite element
$[M_f]$	mass matrix for the whole fluid
$[M_s]$	mass matrix for the whole shell

$[N]$	displacement function defined by equation (10)
$[P]$	elasticity matrix
$[Q]$	defined by equation (11)
$[R]$	defined by equation (6)
$[S_f]$	defined by equation (35)
$[T_m]$	defined by equation (3)
$\{\delta_i\}$	degree of freedom at node i
$\{\varepsilon\}$	deformation vector
$\{\sigma\}$	stress vector